ILLINOIS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Lecture 20: Secret Sharing

CS 539 / ECE 526

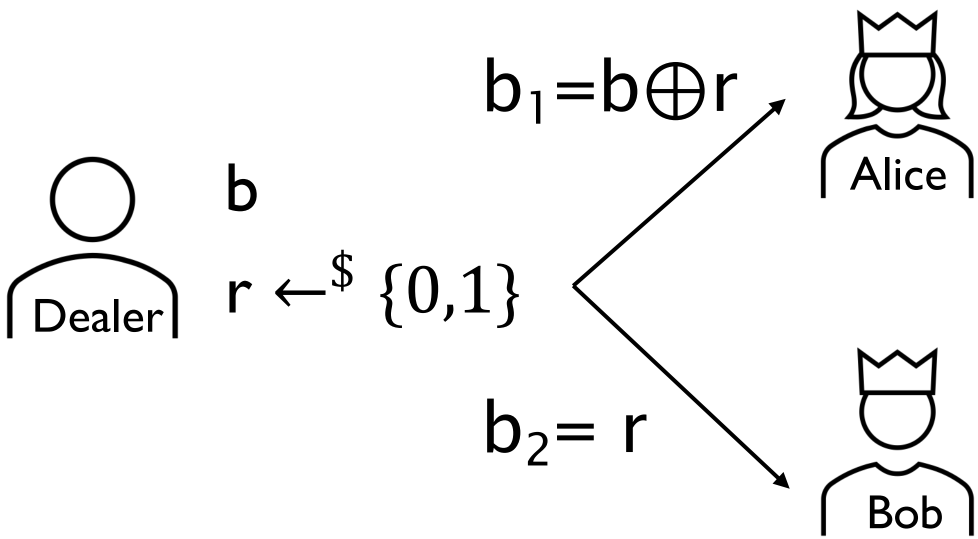
Sourav Das

Secret Sharing

- Activity in groups of 3
- (2, 1) secret sharing for a bit:
  - A dealer shares a secret bit b
  - Each party gets a share (2 parties in total)
  1. Parties jointly can recover b
  2. Share of a single party reveal no information about b
- Hint: One party (party 1) will get a random bit b_1

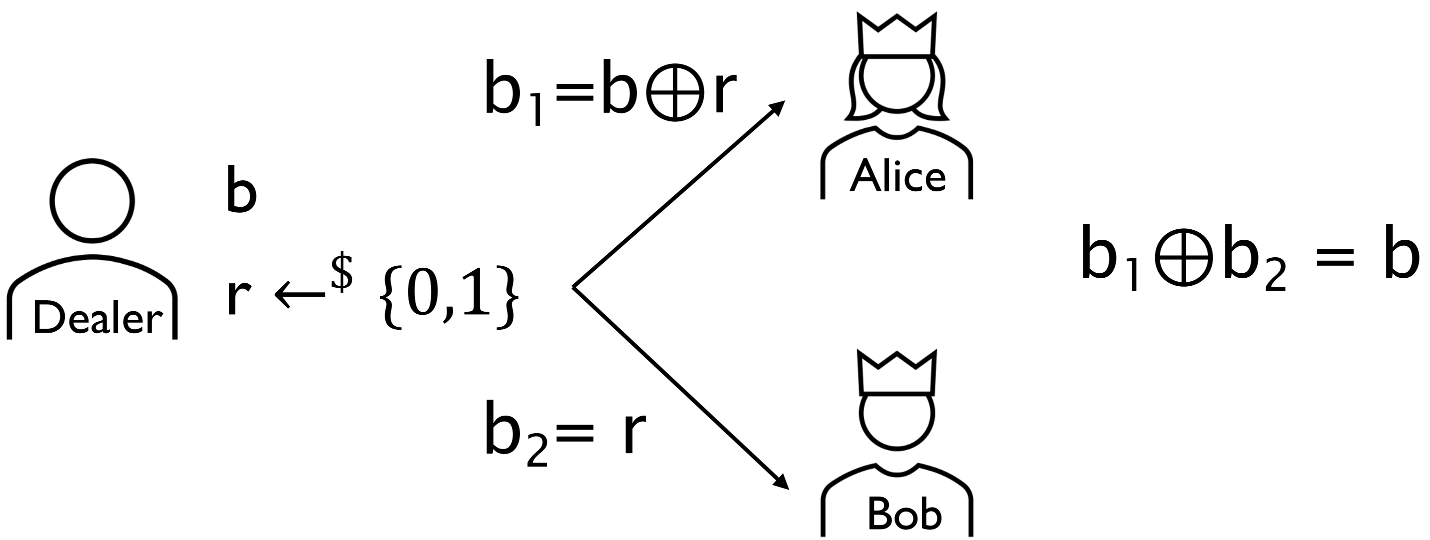
Secret Sharing: Protocol

- (2, 1) secret sharing for a bit:
  - A dealer shares a secret bit b
  - Each party gets a share (2 parties in total)



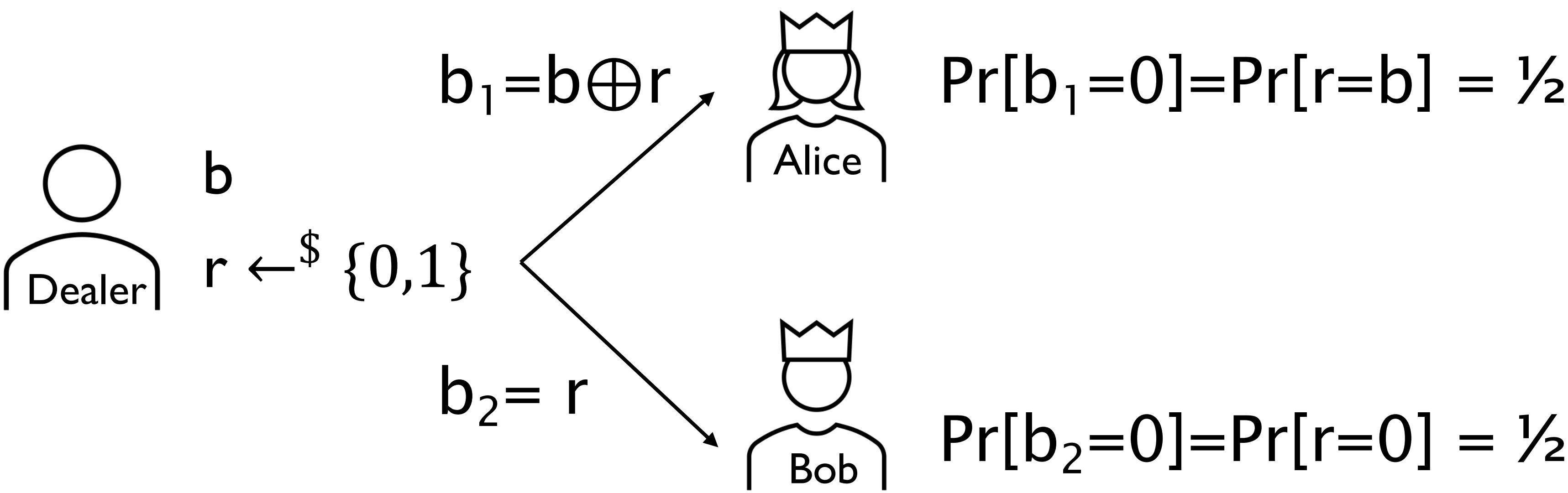
Secret Sharing : Reconstruction

- (2, 1) secret sharing for a bit:
  1. Parties jointly can recover b
  2. Share of a single party reveal no information about b



Secret Sharing : Security

- (2, 1) secret sharing for a bit:
  1. Parties jointly can recover b
  2. Share of a single party reveal no information about b



Secret Sharing

- (n, t) secret sharing:
  - A dealer shares a secret s
  - Each party gets a share (n parties in total)
  - Any t shares reconstruct s
  - Any t-1 shares reveal no information about s
- Tolerate t-1 curious parties and n-t crash faults

Hint 1: Use polynomials of degree t-1
Hint 2: Any t-1 evaluation points does not reveal the entire polynomial

Shamir's Secret Sharing [Shamir 1979]

- $y = f(x) = s + c_1 x + c_2 x^2 + c_2 x^2 + \ldots + c_{t-1} x^{t-1}$
  - $s = f(0)$ is the secret. Other coefficients are random
- Party i's share is $s_i = f(a_i)$
  - $a_1, a_2, a_3, \ldots, a_n$ are distinct public values
- t points fix a degree t-1 polynomial; can reconstruct using Lagrange interpolation

Lagrange Interpolation Formula

Let $(x_1, y_1), \ldots, (x_n, y_n)$ be n points with different x coordinates, then

$$P(x) = \sum_{i=1}^{n} \left(y_i \prod_{j \neq i} \frac{(x - x_j)}{(x_i - x_j)} \right)$$

is the only polynomial of degree $\leq n-1$ that goes through all of them

$$X = \{x_1, x_2, \ldots, x_n\}$$

$$L_{i,X}(x) = \prod_{j \neq i} \frac{(x - x_j)}{(x_i - x_j)}$$

1. Degree of $L_{i,X}(x)$?
2. Value of $L_{i,X}(x_i)$
3. Value of $L_{i,X}(x_j)$ for $j \neq i$

Shamir's Secret Sharing [Shamir 1979]

- $y = f(x) = s + c_1 x + c_2 x^2 + c_2 x^2 + \ldots + c_{t-1} x^{t-1}$
- Will work with polynomials in a finite field
  - All numbers, and + and * operations are mod p where p is a pre-chosen prime
  - Secret $s \in \mathbf{Z}_p = \{0, 1, 2, \ldots, p\text{-}1\}$

Error Correction Codes

- Encode a message m of k symbols into n > k symbols
- Can decode m despite some missing symbols (erasure) or corrupt symbols (error correction)
- Contrast with secret sharing?
- Some simple codes?

Reed-Solomon Code

- $n = k + d$, i.e., d redundancy
- Can tolerate d erasures or $d/2$ errors
- Encode:
  - Chunk msg m as $[m_1, m_2, \ldots, m_k]$ s.t. $m_i \in \mathbf{Z}_p$
  - Find a degree k-1 polynomial $f(x)$ s.t. $f(a_i) = m_i \; \forall \; i \leq k$
  - Compute $f(a_i)$ for $\forall \; k+1 \leq i \leq n$
  - Encoded msg = $[\, f(a_1), f(a_2), \ldots, f(a_n) \,]$

Reed-Solomon Code

- Decode with erasure: Lagrange interpolation!
- Decode with error correction
  - Given $b_1, b_2, \ldots, b_n$ where bi = $f(a_i)$ except d/2 points
  - Let $e(x)$ be an "error locating polynomial", i.e.,

    $$e(a_i) = 0 \quad \text{iff} \quad b_i \neq f(a_i)$$

    - $e(x)$ has $\leq d/2$ distinct roots, hence degree $\leq d/2$
    - We have $e(a_i)\, f(a_i) = e(a_i)\, b_i$
  - Can solve the above system equations!

Reed-Solomon Code

- $e(x)$ has $\le d/2$ distinct roots, hence degree $\le d/2$
- Solve system equations $e(a_i)\ f(a_i) = e(a_i)\ b_i$

– How many unknowns?

- All coefficients of $e()$ and $f()$, so $d/2 + k$

– How many equations?

- n equations but $d/2$ of them are same $(0 = 0)$
- At least $n - d/2 = k + d - d/2 = k + d/2$